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Dynamics of a Strongly Coupled Polaron

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Abstract. We study the dynamics of large polarons described by the Fröhlich Hamiltonian in the limit of strong coupling. The initial conditions are (perturbations of) product states of an electron wave function and a phonon coherent state, as suggested by Pekar. We show that, to leading order on the natural time scale of the problem, the phonon field is stationary and the electron moves according to an effective linear Schrödinger equation.

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1. Introduction and Main Result

The polaron is a model for an electron interacting with the quantized optical modes of a polar crystal. A 'large' (or 'continuous') polaron is characterized by the fact that the spatial extension of this polaron is large compared to the spacing of the underlying lattice. It can be described, as derived by Fröhlich [6] in 1937, by the Hamiltonian

$$H_\alpha^{\mathrm{F}} = p^2 + \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{|k|} \left(\mathrm{e}^{-ik\cdot x} a_k + \mathrm{e}^{ik\cdot x} a_k^* \right) + \int_{\mathbb{R}^3} \mathrm{d}k \, a_k^* a_k,$$

acting in $L^2(\mathbb{R}^3) \otimes \mathcal{F}$. Here, x and $p = -i\nabla_x$ are position and momentum of the electron, respectively, and a_k^* and a_k are creation and annihilation operators in the symmetric Fock space $\mathcal{F}$ over $L^2(\mathbb{R}^3)$, satisfying

$$[a_k, a_{k'}^*] = \alpha^{-2}\delta(k-k'), \quad [a_k, a_{k'}] = [a_k^*, a_{k'}^*] = 0 \quad \text{for all } k,\ k' \in \mathbb{R}^3. \tag{1}$$

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Note the α dependence in the commutation relations. We have written the Hamiltonian in strong coupling units, which will be convenient for us. In the appendix, we explain the change of variables and relate it to the more standard form of this Hamiltonian. In Section 2. we also discuss the precise definition of this Hamiltonian and its lower boundedness.

Through the commutation relations, the Hamiltonian H_α^{F} depends on a single non-negative parameter $\alpha > 0$, and we are interested in the so-called 'strong coupling regime' $\alpha \to \infty$. The ground state energy

$$E_\alpha^{\mathrm{F}} = \inf \operatorname{spec} H_\alpha^{\mathrm{F}}$$

has been studied extensively. While its behavior for small α was understood completely by the middle of the 1950s [5,7,9,10,12] the strong coupling regime remained open for quite some time. Pekar [13,14] had produced an upper bound on E_α^{F} using a trial state of the product form

$$\Psi = \psi \otimes W(\alpha^2 \varphi)\Omega, \tag{2}$$

where $\psi \in H^1(\mathbb{R}^3)$ is the wave function of an electron and $W(\alpha^2\varphi)\Omega$ is a coherent state corresponding to a phonon field $\varphi \in L^2(\mathbb{R}^3)$. More formally, Ω is the vacuum in $\mathcal{F}$ and $W(f)$ is the Weyl operator,

$$W(f) = \exp(a^*(f) - a(f)).$$

For each $f \in L^2(\mathbb{R}^3)$, $W(f)$ is a unitary operator in $\mathcal{F}$. The property of these operators that will be important for us is that

$$W^*(f) a_k W(f) = a_k + \alpha^{-2} f(k) \quad \text{and} \quad W^*(f) a_k^* W(f) = a_k^* + \alpha^{-2}\overline{f(k)}. \tag{3}$$

In particular, coherent states are eigenstates of annihilation operators,

$$a_k W(f)\Omega = \alpha^{-2} f(k) W(f)\Omega. \tag{4}$$

The α enters in (2) so that for fixed ψ and φ, the expected energy is bounded (indeed, constant) with respect to α. To see this, we compute using (4)

$$\left\langle \psi \otimes W(\alpha^2\varphi)\Omega, H_\alpha^{\mathrm{F}} \left(\psi \otimes W(\alpha^2\varphi)\Omega \right) \right\rangle_{L^2(\mathbb{R}^3)\otimes\mathcal{F}} = \langle \psi, H_\varphi \psi \rangle_{L^2(\mathbb{R}^3)} \tag{5}$$

with the effective Schrödinger operator

$$H_\varphi = p^2 + V_\varphi(x) + \|\varphi\|_2^2 \tag{6}$$

acting in $L^2(\mathbb{R}^3)$.

$$V_\varphi(x) = \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{|k|} \left(\mathrm{e}^{-ik\cdot x}\varphi(k) + \mathrm{e}^{ik\cdot x}\overline{\varphi(k)} \right) = 2^{3/2}\pi^{-1/2} \operatorname{Re} \int_{\mathbb{R}^3} \frac{\mathrm{d}x}{|x-x'|^2} \check{\varphi}(x')$$

and

$$\check{\varphi}(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \mathrm{d}k \, \mathrm{e}^{-ik\cdot x} \varphi(k).$$

By minimizing (5) over all ψ and φ, Pekar obtained an upper bound on E_α^{F} which he expected to be asymptotically correct as $\alpha \to \infty$. A mathematically rigorous proof of this fact was only achieved in 1983 by Donsker and Varadhan [4] using large deviation theory; for an alternative proof, using operator theory, see [11].

While the ground state energy E_α^{F} has been studied extensively, we are not aware of any mathematically rigorous study of the dynamics $\mathrm{e}^{-iH_\alpha^{\mathrm{F}} t}\Psi$, although there have been of course several contributions in the physics literature, starting with [5,8] (see [2,3] for more recent results). This is our concern here. More precisely, we are interested in the dynamics in the strong coupling limit $\alpha \to \infty$ for initial date Ψ of the product form (2) suggested by Pekar. Here is a special case of our main result.

THEOREM 1. *Let $\varphi \in L^2(\mathbb{R}^3)$ and $\alpha_0 > 0$. Then for all $\psi \in H^1(\mathbb{R}^3)$, all $\alpha \geq \alpha_0$ and all $t \in \mathbb{R}$,*

$$\left\| \mathrm{e}^{-iH_\alpha^F t} \left(\psi \otimes W(\alpha^2 \varphi)\Omega \right) - \left(\mathrm{e}^{-iH_\varphi t} \psi \right) \otimes W(\alpha^2\varphi)\Omega \right\|^2 \leq 2\alpha^{-2} \|\psi\|^2_{H^1(\mathbb{R}^3)} \left(\mathrm{e}^{C|t|} - 1 \right), \tag{7}$$

where C depends only on α_0 and an upper bound on $\|\varphi\|_2$.

In other words, the evolution of a Pekar product state (2) can be approximated by dynamics of the electron wave function ψ only, and this evolution is described by the Schrödinger operator H_φ in (6) with the effective potential V_φ determined by φ. The coherent state describing the phonon field is stationary. Our main result, Theorem 2, states that this approximation is also valid for certain initial states close to $\psi \otimes W(\alpha^2\varphi)\Omega$ in an appropriate sense.

Observe that (7) establishes the convergence of the full evolution towards the dynamics generated by the Hamiltonian (6) for times $|t| \leq o(\ln \alpha)$ and, in particular, for times of order one. It is natural to ask whether this is the correct time scale to be studied. We believe that this is indeed the case, because, as Theorem 1 shows, on this time scale the system (in particular, the electron) undergoes non-trivial changes. This means that expectations of observables depending on the degrees of freedom of the electron will show a non-trivial evolution. Of course, one can also investigate what happens on different scales. For sufficiently long times $|t| \simeq O(\alpha^\delta)$, with an appropriate $\delta > 0$, we expect the phonon field to exhibit a non-trivial dynamics. In fact, in the physics literature the motion of a strongly coupled polaron is typically described by the non-linear system of equations

$$i\partial_t \psi = (-\Delta + V)\,\psi, \quad \left(c^{-2}\partial_t^2 + 1\right)\Delta V = 4\pi|\psi|^2; \tag{8}$$

see, for instance, [1,3,8]. Our result and the expectation that the phonon field evolves on longer time scales are consistent with (8), if the velocity $c \simeq \alpha^{-\delta}$ depends on α and vanishes in the limit $\alpha \to \infty$.

Let us now state a more general version of Theorem 1 which also allows deviations from an exact product structure. To formulate our assumptions on the initial state, we introduce the number of particles operator

$$\mathcal{N} = \int_{\mathbb{R}^3} \mathrm{d}k\, a_k^* a_k \tag{9}$$

acting in $\mathcal{F}$. Note that, if $\xi = (\xi^{(0)}, \xi^{(1)}, \ldots) \in \mathcal{F}$, then

$$\langle \xi, \mathcal{N}\xi\rangle = \alpha^{-2}\sum_{n=1}^{\infty} n\|\xi^{(n)}\|^2.$$

The factor α^{-2} on the right side comes from the α-dependence of the canonical commutation relations.

Our main result reads as follows.

THEOREM 2. *Let $\varphi \in L^2(\mathbb{R}^3)$ and $\alpha_0 > 0$. Assume that $\Psi \in L^2(\mathbb{R}^3) \otimes \mathcal{F}$ satisfies*

$$\|(p^2 + \mathcal{N} + 1)^{1/2}\Psi\| \le M, \quad \|(p^2+1)^{1/2}\mathcal{N}\Psi\| \le M\alpha^{-2}. \tag{10}$$

Then for all $\alpha \ge \alpha_0$ and all $t \in \mathbb{R}$,

$$\left\| \mathrm{e}^{-iH_\alpha^F t} W(\alpha^2\varphi)\Psi - \mathrm{e}^{-iH_\varphi t} W(\alpha^2\varphi)\Psi \right\|^2 \le 2M^2\alpha^{-2}\left(\mathrm{e}^{C|t|} - 1\right),$$

where C depends only on α_0 and an upper bound on $\|\varphi\|_2$.

This implies, of course, Theorem 1 by taking $\Psi = \psi \otimes \Omega$. Since $\mathcal{N}\Omega = 0$, the two conditions in (10) are satisfied with $M = \|\psi\|_{H^1}$, provided $\psi \in H^1(\mathbb{R}^3)$.

There is nothing special about the constant 2 in this theorem (or in Theorem 1). It can be replaced by any constant greater than one.

We now describe the strategy of our proof. We first observe that, since $W(\alpha^2\varphi)$ is unitary and commutes with H_φ, we have

$$\begin{aligned}\left\| \mathrm{e}^{-iH_\alpha^{\mathrm{F}} t} W(\alpha^2\varphi)\Psi - \mathrm{e}^{-iH_\varphi t} W(\alpha^2\varphi)\Psi \right\|^2 &= \left\| W^*(\alpha^2\varphi)\mathrm{e}^{-iH_\alpha^{\mathrm{F}} t} W(\alpha^2\varphi)\Psi - \mathrm{e}^{-iH_\varphi t}\Psi \right\|^2 \\ &= \left\| \mathrm{e}^{-iW^*(\alpha^2\varphi)H_\alpha^{\mathrm{F}}W(\alpha^2\varphi)t}\Psi - \mathrm{e}^{-iH_\varphi t}\Psi \right\|^2.\end{aligned}$$

Moreover, a short computation based on (3), shows that

$$\begin{aligned} W^*(\alpha^2\varphi)H_\alpha^{\mathrm{F}}W(\alpha^2\varphi) &= H_\varphi + \int_{\mathbb{R}^3} \mathrm{d}k\, a_k^* a_k + a(\varphi) + a^*(\varphi) + \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{|k|}\left(\mathrm{e}^{-ik\cdot x}a_k + \mathrm{e}^{ik\cdot x}a_k^*\right) \\ &=: H. \end{aligned}$$

(Here, for the sake of simplicity, we do not indicate the dependence of H on α and φ.) In Section 2 we shall show that H, and therefore H_α^{F} as well, are lower semi-bounded operators in $L^2(\mathbb{R}^3)\otimes\mathcal{F}$. Since $|k|^{-1}\notin L^2(\mathbb{R}^3)$, this is not completely obvious.

These manipulations have reduced the proof of Theorem 2 to the proof of the bound

$$\left\|\mathrm{e}^{-iHt}\Psi-\mathrm{e}^{-iH_\varphi t}\Psi\right\|^2\leq 2M^2\alpha^{-2}\left(\mathrm{e}^{C|t|}-1\right) \tag{11}$$

with C depending only on α_0 and an upper bound on $\|\varphi\|_2$. We shall prove (11) using a Gronwall-type argument, as explained in Proposition 9.

2. Form Boundedness and Energy Conservation

2.1. THE OPERATOR H_φ

Let $\varphi\in L^2(\mathbb{R}^3)$. We want to argue that the potential $\check{\varphi}*|x|^{-2}$ is infinitesimally form-bounded with respect to the Laplacian. Indeed, by the Hardy–Littlewood–Sobolev inequality $\check{\varphi}*|x|^{-2}\in L^6(\mathbb{R}^3)$ and therefore, by Hölder's inequality,

$$\int_{\mathbb{R}^d}|\check{\varphi}*|x|^{-2}||\psi|^2\,\mathrm{d}x\leq\|\check{\varphi}*|x|^{-2}\|_6\|\psi\|_{12/5}^2\leq\|\check{\varphi}*|x|^{-2}\|_6\|\psi\|_6^{1/2}\|\psi\|_2^{3/2}.$$

By Sobolev's inequality, we conclude that there is a C such that for every $\varepsilon>0$,

$$\int_{\mathbb{R}^d}|\check{\varphi}*|x|^{-2}||\psi|^2\,\mathrm{d}x\leq\varepsilon(\psi,p^2\psi)+C\varepsilon^{-1/3}\|\check{\varphi}*|x|^{-2}\|_6^{4/3}\|\psi\|_2^2.$$

Thus, $\check{\varphi}*|x|^{-2}$ is infinitesimally form-bounded with respect to p^2, and we have

$$H_\varphi\geq(1-\varepsilon)p^2-C_\varepsilon\quad\text{and}\quad H_\varphi\leq(1+\varepsilon)p^2+C_\varepsilon \tag{12}$$

with $C_\varepsilon=\varepsilon^{-1/3}\|\check{\varphi}*|x|^{-2}\|_6^{4/3}+\|\varphi\|_2^2$. These two bounds imply (almost) conservation of the kinetic energy.

LEMMA 3. *If $\varphi\in L^2(\mathbb{R}^3)$, then*

$$\sup_{t\in\mathbb{R}}\|(p^2+1)^{1/2}\mathrm{e}^{-iH_\varphi t}(p^2+1)^{-1/2}\|<\infty.$$

Proof. For $\psi\in H^1(\mathbb{R}^3)$, by (12),

$$\begin{aligned}\||p|\mathrm{e}^{-iH_\varphi t}\psi\|_2^2&\leq(1-\varepsilon)^{-1}\|(H_\varphi+C_\varepsilon)^{1/2}\mathrm{e}^{-iH_\varphi t}\psi\|_2^2\\&=(1-\varepsilon)^{-1}\|(H_\varphi+C_\varepsilon)^{1/2}\psi\|_2^2\\&\leq(1-\varepsilon)^{-1}\|((1+\varepsilon)p^2+2C_\varepsilon)^{1/2}\psi\|_2^2\end{aligned}$$

This clearly implies the assertion. □

2.2. CREATION AND ANNIHILATION OPERATORS

In this section, we consider operators of the form

$$a(\mathrm{e}^{ik\cdot x}f)=\int_{\mathbb{R}^3}\mathrm{d}k\,\mathrm{e}^{-ik\cdot x}\overline{f(k)}a_k \quad\text{and}\quad a^*(\mathrm{e}^{ik\cdot x}f)=\int_{\mathbb{R}^3}\mathrm{d}k\,\mathrm{e}^{ik\cdot x}f(k)a_k^*$$

acting in $L^2(\mathbb{R}^3)\otimes\mathcal{F}$, where f is a given function in $L^2(\mathbb{R}^3)$. We shall show that these operators can be bounded in terms of the square root of the number of particles operator $\mathcal{N}$, see (9). We have

LEMMA 4. *Let* $f\in L^2(\mathbb{R}^3)$. *Then*

$$\left\|a(\mathrm{e}^{ik\cdot x}f)\Psi\right\|\leq\|f\|_2\left\|\mathcal{N}^{1/2}\Psi\right\| \quad\textit{and}\quad \left\|a^*(\mathrm{e}^{ik\cdot x}f)\Psi\right\|\leq\|f\|_2\left\|(\mathcal{N}+\alpha^{-2})^{1/2}\Psi\right\|.$$

Moreover,

$$\left\|\mathcal{N}^{1/2}a(\mathrm{e}^{ik\cdot x}f)\Psi\right\|\leq\|f\|_2\left\|\left(\mathcal{N}\left(\mathcal{N}-\alpha^{-2}\right)\right)^{1/2}\Psi\right\|$$

and

$$\left\|\mathcal{N}^{1/2}a^*(\mathrm{e}^{ik\cdot x}f)\Psi\right\|\leq\|f\|_2\left\|\left(\mathcal{N}+\alpha^{-2}\right)\Psi\right\|.$$

The proof is well known and elementary, but we include it for the sake of completeness.

Proof. The first inequality follows from

$$\left\|a(\mathrm{e}^{ik\cdot x}f)\Psi\right\|\leq\int_{\mathbb{R}^3}\mathrm{d}k\,|f(k)|\|a_k\Psi\|\leq\|f\|_2\left\|\mathcal{N}^{1/2}\Psi\right\|.$$

To prove the second one, we use the intertwining relations

$$a(f)h(\mathcal{N})=h(\mathcal{N}+\alpha^{-2})a(f) \quad\text{and}\quad a^*(f)h(\mathcal{N}+\alpha^{-2})=h(\mathcal{N})a^*(f), \tag{13}$$

which hold for any function $h:\alpha^{-2}\mathbb{N}_0\to\alpha^{-2}\mathbb{N}_0$ and follow from the canonical commutation relations (1). These relations (together with the first bound in the lemma) imply that

$$\|a^*(\mathrm{e}^{ik\cdot x}f)(\mathcal{N}+\alpha^{-2})^{-1/2}\|=\|\mathcal{N}^{-1/2}a^*(\mathrm{e}^{ik\cdot x}f)\|=\|a(\mathrm{e}^{ik\cdot x}f)\mathcal{N}^{-1/2}\|\leq\|f\|_2.$$

The third and fourth bound follow from the first two and again from the intertwining relations (13). □

We shall need the following corollary later in our proof.

COROLLARY 5. *Let* $(1+|k|)f \in L^2(\mathbb{R}^3)$. *Then*

$$\left\| \left(p^2+1\right)^{1/2} \mathcal{N}^{1/2} a^*(\mathrm{e}^{ik\cdot x} f)\Psi \right\| \leq C \,\|(1+|k|)\, f\|_2 \left\| \left(p^2+1\right)^{1/2} \left(\mathcal{N}+\alpha^{-2}\right) \Psi \right\|.$$

Proof. We write

$$\begin{aligned} &\left\| \left(p^2+1\right)^{1/2} \mathcal{N}^{1/2} a^*(\mathrm{e}^{ik\cdot x} f)\Psi \right\|^2 \\ &\quad = \|\mathcal{N}^{1/2} a^*(\mathrm{e}^{ik\cdot x} f)\Psi\|^2 + \sum_{j=1}^{3} \left\| \mathcal{N}^{1/2} p_j a^*(\mathrm{e}^{ik\cdot x} f)\Psi \right\|^2. \end{aligned}$$

The bound for $\mathcal{N}^{1/2} a^*(\mathrm{e}^{ik\cdot x} f)\Psi$ follows from the second part of Lemma 4. To bound the remaining terms, we observe that

$$p_j a^*(\mathrm{e}^{ik\cdot x} f) = a^*(\mathrm{e}^{ik\cdot x} f) p_j + [p_j, a^*(\mathrm{e}^{ik\cdot x} f)] = a^*(\mathrm{e}^{ik\cdot x} f) p_j + a^*(\mathrm{e}^{ik\cdot x} k_j f).$$

Thus,

$$\left\| \mathcal{N}^{1/2} p_j a^*(\mathrm{e}^{ik\cdot x} f)\Psi \right\| \leq \left\| \mathcal{N}^{1/2} a^*(\mathrm{e}^{ik\cdot x} f) p_j \Psi \right\| + \left\| \mathcal{N}^{1/2} a^*(\mathrm{e}^{ik\cdot x} k_j f)\Psi \right\|$$

and the assertion follows again from the second part of Lemma 4. □

2.3. THE OPERATOR H

Our next goal is to prove that the operator H is lower semi-bounded. Indeed, we shall show that H differs form $p^2+\mathcal{N}$ by terms which are infinitesimally form-bounded with respect to $p^2+\mathcal{N}$. We begin with

LEMMA 6. *If* $f \in L^2(\mathbb{R}^3)$ *and* $\varepsilon > 0$, *then*

$$a(\mathrm{e}^{ikx} f) + a^*(\mathrm{e}^{ikx} f) \leq \varepsilon \mathcal{N} + \varepsilon^{-1} \|f\|_2^2.$$

Clearly, replacing f by $-f$, we also obtain

$$a(\mathrm{e}^{ikx} f) + a^*(\mathrm{e}^{ikx} f) \geq -\varepsilon \mathcal{N} - \varepsilon^{-1} \|f\|_2^2.$$

Proof. We have

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^3} \mathrm{d}k \left(\varepsilon^{1/2} a_k^* - \varepsilon^{-1/2} \mathrm{e}^{-ikx} \overline{f(k)}\right) \left(\varepsilon^{1/2} a_k - \varepsilon^{-1/2} \mathrm{e}^{ikx} f(k)\right) \\ &= \varepsilon \mathcal{N} + \varepsilon^{-1} \|f\|_2^2 - a^*(\mathrm{e}^{ikx} f) - a(\mathrm{e}^{ikx} f), \end{aligned}$$

which implies the assertion. □

The following lemma is considerably more involved. It allows one to deal with the non-L^2 tail of $|k|^{-1}$, and is essentially due to Lieb and Yamazaki [12].

LEMMA 7. *If* $|k|^{-1} f \in L^2(\mathbb{R}^3)$ *and* $\varepsilon > 0$*, then*

$$a(\mathrm{e}^{ikx} f) + a^*(\mathrm{e}^{ikx} f) \le \varepsilon p^2 + 2\varepsilon^{-1} \| |k|^{-1} f\|_2^2 \left(2\mathcal{N} + \alpha^{-2}\right).$$

Again, replacing f by $-f$, we obtain

$$a(\mathrm{e}^{ikx} f) + a^*(\mathrm{e}^{ikx} f) \ge -\varepsilon p^2 - 2\varepsilon^{-1} \| |k|^{-1} f\|_2^2 \left(2\mathcal{N} + \alpha^{-2}\right).$$

Proof. For $j = 1, 2, 3$, we introduce

$$Z_j = \int_{\mathbb{R}^d} \mathrm{d}k \, \frac{k_j}{k^2} \, \mathrm{e}^{-ikx} \, \overline{f(k)} a_k$$

and write

$$a(\mathrm{e}^{ikx} f) + a^*(\mathrm{e}^{ikx} f) = \sum_{j=1}^{3} [Z_j - Z_j^*, p_j] = \sum_{j=1}^{3} \left(\left(Z_j - Z_j^*\right) p_j + p_j \left(Z_j^* - Z_j\right)\right).$$

We bound, for every j,

$$\begin{aligned}
\left(Z_j - Z_j^*\right) p_j + p_j \left(Z_j^* - Z_j\right) &\le \varepsilon p_j^2 + \varepsilon^{-1} \left(Z_j - Z_j^*\right)\left(Z_j^* - Z_j\right) \\
&\le \varepsilon p_j^2 + 2\varepsilon^{-1} \left(Z_j^* Z_j + Z_j Z_j^*\right) \\
&= \varepsilon p_j^2 + 2\varepsilon^{-1} \left(2 Z_j^* Z_j + [Z_j, Z_j^*]\right).
\end{aligned}$$

It remains to bound the last two terms. For every Ψ, we have, by Cauchy–Schwarz,

$$\begin{aligned}
\left\langle \Psi, \sum_{j=1}^{3} Z_j^* Z_j \Psi \right\rangle &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\mathrm{d}k'}{k'^2} \frac{\mathrm{d}k}{k^2} k' \cdot k \;\; f(k') \overline{f(k)} \langle \Psi, a_{k'}^* \mathrm{e}^{i(k'-k)\cdot x} a_k \Psi \rangle \\
&\le \left(\int_{\mathbb{R}^3} \frac{\mathrm{d}k}{|k|} |f(k)| \|a_k \Psi\| \right)^2 \\
&\le \| |k|^{-1} f\|_2^2 \langle \Psi, \mathcal{N} \Psi \rangle.
\end{aligned}$$

On the other hand, because of the commutation relations we have

$$\sum_{j=1}^{3} [Z_j, Z_j^*] = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\mathrm{d}k'}{k'^2} \frac{\mathrm{d}k}{k^2} k' \cdot k \;\; \overline{f(k')} f(k) \mathrm{e}^{-i(k'-k)\cdot x} [a_{k'}, a_k^*] = \alpha^{-2} \| |k|^{-1} f\|_2^2.$$

This concludes the proof of the lemma. □

We are now in position to prove form boundedness. Given a number $\Lambda > 0$ to be specified later, we decompose

$$H = H_\varphi + A + B + B^*, \tag{14}$$

where

$$A = \mathcal{N} + a(\varphi) + a^*(\varphi) + \int\limits_{|k|<\Lambda} \frac{\mathrm{d}k}{|k|} \left(\mathrm{e}^{ik\cdot x} a_k + \mathrm{e}^{-ik\cdot x} a_k^*\right) \tag{15}$$

and

$$B = \int\limits_{|k|>\Lambda} \frac{\mathrm{d}k}{|k|} \mathrm{e}^{ik\cdot x} a_k. \tag{16}$$

For any choice of $\Lambda > 0$, Lemma 6 implies that $A - \mathcal{N}$ is infinitesimally form-bounded with respect to $\mathcal{N}$.

We claim that for any $\varepsilon > 0$ there is a $\Lambda > 0$ such that $B + B^*$ is form-bounded with respect to $p^2 + \mathcal{N}$ with form bound ε. Indeed, this follows from Lemma 7 by choosing Λ so large that

$$4\varepsilon^{-1} \left\| |k|^{-2} \chi_{\{|k|>\Lambda\}} \right\|_2^2 = \varepsilon.$$

This argument shows that for every $\varepsilon > 0$ there is a C_ε such that

$$H \geq (1-\varepsilon)(p^2 + \mathcal{N}) - C_\varepsilon \quad \text{and} \quad H \leq (1+\varepsilon)(p^2 + \mathcal{N}) + C_\varepsilon.$$

The constant C_ε depends on α through the use of Lemma 7, but it is uniformly bounded for $\alpha \geq \alpha_0$. Thus, by the same argument as in Lemma 3 we obtain

LEMMA 8. *If $\varphi \in L^2(\mathbb{R}^3)$ and $\alpha_0 > 0$, then*

$$\sup_{\alpha \geq \alpha_0} \sup_{t \in \mathbb{R}} \left\| \left(p^2 + \mathcal{N} + 1\right)^{1/2} \mathrm{e}^{-iHt} \left(p^2 + \mathcal{N} + 1\right)^{-1/2} \right\| < \infty.$$

3. Proof of Theorem 2

We shall prove Theorem 2 by a Gronwall-type argument. More precisely, we shall prove the following proposition.

PROPOSITION 9. *Let Ψ be as in Theorem 2. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\| \left(\mathrm{e}^{-iHt} - \mathrm{e}^{-iH_\varphi t}\right) \Psi \right\|^2 = f(t) + g(t),$$

where

$$f(t) \leq CM\alpha^{-1} \left\| \left(\mathrm{e}^{-iHt} - \mathrm{e}^{-iH_\varphi t}\right) \Psi \right\|$$

and, for all $T \geq 0$,

$$\int_0^T \mathrm{d}t\, g(t) \leq C M^2 \alpha^{-2} T.$$

Here, C depends only on α_0 *and an upper bound on* $\|\varphi\|_2$.

Proof of Theorem 2 given Proposition 9. It suffices to consider times $T \geq 0$. Then

$$A(T) := \left\| \left(\mathrm{e}^{-iHT} - \mathrm{e}^{-iH_\varphi T} \right) \Psi \right\|^2 = \int_0^T \mathrm{d}t\, f(t) + \int_0^T \mathrm{d}t\, g(t).$$

According to Proposition 9, we have $f(t) \leq CM^2\alpha^{-2} + CA(t)$. This, together with the bound on the integral of g, implies

$$A(T) \leq 2CM^2\alpha^{-2}T + C \int_0^T \mathrm{d}t\, A(t).$$

Thus,

$$\left(A(T) + 2M^2\alpha^{-2} \right) \leq 2M^2\alpha^{-2} + C \int_0^T \mathrm{d}t\, (A(t) + 2M^2\alpha^{-2})$$

and, by Gronwall's inequality,

$$A(t) + 2M^2\alpha^{-2} \leq 2M^2\alpha^{-2}\mathrm{e}^{Ct}.$$

This is inequality (11) which, as explained before, is equivalent to the inequality stated in Theorem 2.

It remains to prove Proposition 9, and so we differentiate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \left\| \left(\mathrm{e}^{-iHt} - \mathrm{e}^{-iH_\varphi t} \right) \Psi \right\|^2 &= 2\,\mathrm{Im} \left\langle \mathrm{e}^{-iHt}\Psi, (H - H_\varphi)\mathrm{e}^{-iH_\varphi t}\Psi \right\rangle \\
&= 2\,\mathrm{Im} \left\langle \left(\mathrm{e}^{-iHt} - \mathrm{e}^{-iH_\varphi t} \right) \Psi, (H - H_\varphi)\mathrm{e}^{-iH_\varphi t}\Psi \right\rangle \\
&= f_1(t) + f_2(t) + h(t).
\end{aligned}$$

In the middle equality, we used the fact that H and H_φ are self-adjoint. The functions f_1, f_2 and h are defined by

$$\begin{aligned}
f_1(t) &= 2\,\mathrm{Im} \left\langle \left(\mathrm{e}^{-iHt} - \mathrm{e}^{-iH_\varphi t} \right) \Psi, A\mathrm{e}^{-iH_\varphi t}\Psi \right\rangle, \\
f_2(t) &= 2\,\mathrm{Im} \left\langle \left(\mathrm{e}^{-iHt} - \mathrm{e}^{-iH_\varphi t} \right) \Psi, B\mathrm{e}^{-iH_\varphi t}\Psi \right\rangle, \\
h(t) &= 2\,\mathrm{Im} \left\langle \left(\mathrm{e}^{-iHt} - \mathrm{e}^{-iH_\varphi t} \right) \Psi, B^*\mathrm{e}^{-iH_\varphi t}\Psi \right\rangle
\end{aligned}$$

in terms of the decomposition $H = H_\varphi + A + B + B^*$ from (14). As we will see below, the functions f_1 and f_2 contribute to the f-piece in Proposition 9, whereas h will be further decomposed into an f-piece and a g-piece.

In the decomposition above, the cut-off value Λ is fixed and we do not make it explicit in our bounds. Also, we do not indicate the dependence of the constants on φ (and its L^2-norm) and α_0. As a final preliminary, let us note that the a-priori bounds (10) imply

$$\left\| \left(p^2+1\right)^{1/2} \mathcal{N}^{1/2}\Psi \right\| \le M\alpha^{-1}. \tag{17}$$

Indeed, this follows by the Cauchy–Schwarz inequality, since $\|(p^2+1)^{1/2}\Psi\| \le M$ and $\|(p^2+1)^{1/2}\mathcal{N}\Psi\| \le M\alpha^{-2}$. Moreover, by (10)

$$\|\mathcal{N}\Psi\| \le CM\alpha^{-1}. \tag{18}$$

with $C = \alpha_0^{-1}$.

3.1. BOUND ON f_1

Recall from (15) the definition of the operator A. It is an easy consequence of Lemma 4 that

$$\|A\xi\| \le C \left\| \left(\mathcal{N} + \alpha^{-1}\right)\xi \right\| \qquad \text{for all } \xi,$$

and, thus,

$$\begin{aligned} |f_1(t)| &\le 2 \left\| \left(\mathrm{e}^{-iHt} - \mathrm{e}^{-iH_\varphi t}\right)\Psi \right\| \left\| A\mathrm{e}^{-iH_\varphi t}\Psi \right\| \\ &\le 2C \left\| \left(\mathrm{e}^{-iHt} - \mathrm{e}^{-iH_\varphi t}\right)\Psi \right\| \left\| \left(\mathcal{N} + \alpha^{-1}\right)\Psi \right\| \\ &\le C' M\alpha^{-1} \left\| \left(\mathrm{e}^{-iHt} - \mathrm{e}^{-iH_\varphi t}\right)\Psi \right\|. \end{aligned}$$

Here, we also used (18) and the fact that $\mathcal{N}$ commutes with H_φ. This bound on f_1 is already of the form required for the application of Proposition 9.

3.2. BOUND ON f_2

To estimate f_2, we make use of the following lemma.

LEMMA 10. *Recall from* (16) *the definition of the operator B. We have, with a constant depending only on Λ,*

$$\|B\xi\| \le C \left\| \left(p^2+1\right)^{1/2} \mathcal{N}^{1/2}\xi \right\|$$

and

$$\left\| \left(\mathcal{N} + \alpha^{-2} \right)^{-1/2} B\xi \right\| \leq C \left\| \left(p^2 + 1 \right)^{1/2} \xi \right\|.$$

Proof. If we describe the electron in momentum space, then $B\xi$ for $\xi = (\xi^{(0)}, \xi^{(1)}, \ldots)$ is given by

$$(B\xi)^{(n)}(p, k_1, \ldots, k_n) = \sqrt{\alpha}\,\sqrt{n+1} \int\limits_{|k|>\Lambda} \frac{\mathrm{d}k}{|k|} \xi^{(n+1)}(p+k, \alpha k, k_1, \ldots, k_n).$$

This follows from the standard representation of $a(f)$ together with the rescaling explained in the "Appendix". By Cauchy–Schwarz,

$$\begin{aligned}
\|B\xi\|^2 &= \alpha \sum_{n=0}^{\infty} (n+1) \int\limits_{\mathbb{R}^3} \mathrm{d}p \int\limits_{\mathbb{R}^{3n}} \mathrm{d}\mathbf{k} \left| \int\limits_{|k|>\Lambda} \frac{\mathrm{d}k}{|k|} \xi^{(n+1)}(p+k, \alpha k, \mathbf{k}) \right|^2 \\
&= \alpha \sum_{n=0}^{\infty} (n+1) \int\limits_{\mathbb{R}^3} \mathrm{d}p \int\limits_{\mathbb{R}^{3n}} \mathrm{d}\mathbf{k} \\
&\quad \times \iint\limits_{|k|>\Lambda, |k'|>\Lambda} \frac{\mathrm{d}k'}{|k'|} \frac{\mathrm{d}k}{|k|} \overline{\xi^{(n+1)}(p+k', \alpha k', \mathbf{k})} \xi^{(n+1)}(p+k, \alpha k, \mathbf{k}) \\
&\leq \alpha \sum_{n=0}^{\infty} (n+1) \int\limits_{\mathbb{R}^3} \mathrm{d}p \int\limits_{\mathbb{R}^{3n}} \mathrm{d}\mathbf{k} \\
&\quad \times \iint\limits_{|k|>\Lambda, |k'|>\Lambda} \mathrm{d}k'\, \mathrm{d}k\, \frac{1+(p+k)^2}{k'^2(1+(p+k')^2)} |\xi^{(n+1)}(p+k, \alpha k, \mathbf{k})|^2 \\
&\leq C\alpha \sum_{n=0}^{\infty} (n+1) \int\limits_{\mathbb{R}^3} \mathrm{d}p \int\limits_{\mathbb{R}^{3n}} \mathrm{d}\mathbf{k} \int\limits_{\mathbb{R}^3} \mathrm{d}k\, (1+(p+k)^2) |\xi^{(n+1)}(p+k, \alpha k, \mathbf{k})|^2 \\
&= C\alpha^{-2} \sum_{n=0}^{\infty} (n+1) \int\limits_{\mathbb{R}^3} \mathrm{d}p \int\limits_{\mathbb{R}^{3n}} \mathrm{d}\mathbf{k} \int\limits_{\mathbb{R}^3} \mathrm{d}\tilde{k}\, (1+p^2) |\xi^{(n+1)}(p, \tilde{k}, \mathbf{k})|^2 \\
&= C \left\| \mathcal{N}^{1/2} (1+p^2)^{1/2} \xi \right\|^2
\end{aligned}$$

with

$$C = \sup_{p \in \mathbb{R}^3} \int\limits_{|k'|>\Lambda} \frac{\mathrm{d}k'}{k'^2(1+(p+k')^2)} < \infty.$$

This proves the first bound in the lemma. The second one is proved similarly and we omit the details. □

Using this lemma, we bound

$$
\begin{aligned}
|f_2(t)| &\le 2\left\|\left(\mathrm{e}^{-iHt}-\mathrm{e}^{-iH_\varphi t}\right)\Psi\right\|\left\|B\mathrm{e}^{-iH_\varphi t}\Psi\right\| \\
&\le 2C\left\|\left(\mathrm{e}^{-iHt}-\mathrm{e}^{-iH_\varphi t}\right)\Psi\right\|\left\|(p^2+1)^{1/2}\mathcal{N}^{1/2}\mathrm{e}^{-iH_\varphi t}\Psi\right\|.
\end{aligned}
$$

Since $\mathcal{N}$ commutes with H_φ, by means of the energy conservation Lemma 3 and by (17) we find

$$
\left\|(p^2+1)^{1/2}\mathcal{N}^{1/2}\mathrm{e}^{-iH_\varphi t}\Psi\right\| \le C'\left\|(p^2+1)^{1/2}\mathcal{N}^{1/2}\Psi\right\| \le C'M\alpha^{-1}.
$$

Thus,

$$
|f_2(t)| \le 2CC'M\alpha^{-1}\left\|\left(\mathrm{e}^{-iHt}-\mathrm{e}^{-iH_\varphi t}\right)\Psi\right\|,
$$

which is a bound of the form required for Proposition 9.

3.3. DECOMPOSITION OF h

It remains to deal with the term h, which involves the operator B^* (recall from (16) the definition of the operator B). We split this operator as follows:

$$
\begin{aligned}
B^* &= \int_{|k|>\Lambda} \frac{\mathrm{d}k}{|k|^3}\left[k\cdot p, \mathrm{e}^{ik\cdot x}\right]a_k^* \\
&= \int_{|k|>\Lambda} \frac{\mathrm{d}k}{|k|^3}\left(k\cdot p\mathrm{e}^{ik\cdot x}+\mathrm{e}^{ik\cdot x}k\cdot p\right)a_k^* - 2\int_{|k|>\Lambda}\frac{\mathrm{d}k}{|k|^3}\mathrm{e}^{ik\cdot x}k\cdot p\ a_k^* \\
&= \left[H_\varphi, a^*\left(\mathrm{e}^{ikx}|k|^{-3}\chi_{\{|k|>\Lambda\}}\right)\right] - 2\int_{|k|>\Lambda}\frac{\mathrm{d}k}{|k|^3}\mathrm{e}^{ik\cdot x}k\cdot p\ a_k^*.
\end{aligned}
$$

Accordingly, we decompose

$$
h(t) = f_3(t) + g(t),
$$

where

$$
f_3(t) = -4\,\mathrm{Im}\left\langle\left(\mathrm{e}^{-iHt}-\mathrm{e}^{-iH_\varphi t}\right)\Psi, \int_{|k|>\Lambda}\frac{\mathrm{d}k}{|k|^3}\mathrm{e}^{ik\cdot x}k\cdot p\ a_k^*\ \mathrm{e}^{-iH_\varphi t}\Psi\right\rangle
$$

and

$$
g(t) = 2\,\mathrm{Im}\left\langle\left(\mathrm{e}^{-iHt}-\mathrm{e}^{-iH_\varphi t}\right)\Psi, \left[H_\varphi, a^*\left(\mathrm{e}^{ikx}|k|^{-3}\chi_{\{|k|>\Lambda\}}\right)\right]\mathrm{e}^{-iH_\varphi t}\Psi\right\rangle.
$$

3.4. BOUND ON f_3

We bound

$$|f_3(t)| \leq 4 \left\| \left(\mathrm{e}^{-iHt} - \mathrm{e}^{-iH_\varphi t}\right) \Psi \right\| \left\| \int\limits_{|k|>\Lambda} \frac{\mathrm{d}k}{|k|^3} \mathrm{e}^{ik\cdot x} k \cdot p \; a_k^* \; \mathrm{e}^{-iH_\varphi t} \Psi \right\|$$
$$\leq 4 \left\| \left(\mathrm{e}^{-iHt} - \mathrm{e}^{-iH_\varphi t}\right) \Psi \right\| \sum_{j=1}^{3} \left\| a^*(\mathrm{e}^{ik\cdot x} k_j |k|^{-3} \chi_{\{|k|>\Lambda\}}) p_j \; \mathrm{e}^{-iH_\varphi t} \Psi \right\| .$$

According to Lemma 4 and energy conservation, Lemma 3, we have

$$\begin{aligned} &\left\| a^*(\mathrm{e}^{ik\cdot x} k_j |k|^{-3} \chi_{\{|k|>\Lambda\}}) p_j \; \mathrm{e}^{-iH_\varphi t} \Psi \right\| \\ &\quad \leq \| k_j |k|^{-3} \chi_{\{|k|>\Lambda\}} \| \left\| \left(\mathcal{N} + \alpha^{-2}\right)^{1/2} p_j \; \mathrm{e}^{-iH_\varphi t} \Psi \right\| \\ &\quad = \| k_j |k|^{-3} \chi_{\{|k|>\Lambda\}} \| \left\| p_j \; \mathrm{e}^{-iH_\varphi t} \left(\mathcal{N} + \alpha^{-2}\right)^{1/2} \Psi \right\| \\ &\quad \leq C \| k_j |k|^{-3} \chi_{\{|k|>\Lambda\}} \| \left\| (p^2+1)^{1/2} \left(\mathcal{N} + \alpha^{-2}\right)^{1/2} \Psi \right\| \\ &\quad \leq \sqrt{2}\, C \| k_j |k|^{-3} \chi_{\{|k|>\Lambda\}} \| M \alpha^{-1}. \end{aligned}$$

Here we used (17). Thus, f_3 is bounded as required for Proposition 9.

3.5. DECOMPOSITION OF THE INTEGRAL OF g

We want to use the fact that

$$\mathrm{e}^{iH_\varphi t} \left[H_\varphi, a^*(\mathrm{e}^{ikx} |k|^{-3} \chi_{\{|k|>\Lambda\}}) \right] \mathrm{e}^{-iH_\varphi t} = -i \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{iH_\varphi t} a^*(\mathrm{e}^{ikx} |k|^{-3} \chi_{\{|k|>\Lambda\}}) \mathrm{e}^{-iH_\varphi t} \right).$$

This implies that

$$\begin{aligned} &\left(\mathrm{e}^{iHt} - \mathrm{e}^{iH_\varphi t}\right) \left[H_\varphi, a^* \left(\mathrm{e}^{ikx} |k|^{-3} \chi_{\{|k|>\Lambda\}} \right) \right] \mathrm{e}^{-iH_\varphi t} \\ &\quad = -i \left(\mathrm{e}^{iHt} - \mathrm{e}^{iH_\varphi t}\right) \mathrm{e}^{-iH_\varphi t} \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{iH_\varphi t} a^* \left(\mathrm{e}^{ikx} |k|^{-3} \chi_{\{|k|>\Lambda\}} \right) \mathrm{e}^{-iH_\varphi t} \right) \\ &\quad = \int_0^t \mathrm{d}s \, \mathrm{e}^{iHs} (H - H_\varphi) \mathrm{e}^{-iH_\varphi s} \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{iH_\varphi t} a^* \left(\mathrm{e}^{ikx} |k|^{-3} \chi_{\{|k|>\Lambda\}} \right) \mathrm{e}^{-iH_\varphi t} \right). \end{aligned}$$

Integrating by parts, we find that

$$\begin{aligned}
&\int_0^T \mathrm{d}t\, g(t)\\
&\quad = 2\,\mathrm{Im}\int_0^T \mathrm{d}t\,\left\langle \int_0^t \mathrm{d}s\, \mathrm{e}^{iH_\varphi s}(H-H_\varphi)\mathrm{e}^{-iHs}\Psi,\ \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{iH_\varphi t}a^*\big(\mathrm{e}^{ikx}|k|^{-3}\chi_{\{|k|>\Lambda\}}\big)\mathrm{e}^{-iH_\varphi t}\Psi\right\rangle\\
&\quad = -2\,\mathrm{Im}\int_0^T \mathrm{d}t\,\left\langle \mathrm{e}^{iH_\varphi t}(H-H_\varphi)\mathrm{e}^{-iHt}\Psi,\ \mathrm{e}^{iH_\varphi t}a^*\big(\mathrm{e}^{ikx}|k|^{-3}\chi_{\{|k|>\Lambda\}}\big)\mathrm{e}^{-iH_\varphi t}\Psi\right\rangle\\
&\qquad + 2\,\mathrm{Im}\left\langle \int_0^T \mathrm{d}s\, \mathrm{e}^{iH_\varphi s}(H-H_\varphi)\mathrm{e}^{-iHs}\Psi,\ \mathrm{e}^{iH_\varphi T}a^*\big(\mathrm{e}^{ikx}|k|^{-3}\chi_{\{|k|>\Lambda\}}\big)\mathrm{e}^{-iH_\varphi T}\Psi\right\rangle\\
&\quad = -2\,\mathrm{Im}\int_0^T \mathrm{d}t\,\left\langle (H-H_\varphi)\mathrm{e}^{-iHt}\Psi,\ \Psi(t)\right\rangle
\end{aligned}$$

with

$$\Psi(t) = \Big(a^*\big(\mathrm{e}^{ikx}|k|^{-3}\chi_{\{|k|>\Lambda\}}\big) - \mathrm{e}^{iH_\varphi(T-t)}a^*\big(\mathrm{e}^{ikx}|k|^{-3}\chi_{\{|k|>\Lambda\}}\big)\mathrm{e}^{-iH_\varphi(T-t)}\Big)\,\mathrm{e}^{-iH_\varphi t}\Psi.$$

We decompose again $H = H_\varphi + A + B + B^*$ as in (14) and accordingly

$$\int_0^T \mathrm{d}t\, g(t) = G_1(T) + G_2(T) + G_3(T)$$

with

$$\begin{aligned}
G_1(T) &= -2\,\mathrm{Im}\int_0^T \mathrm{d}t\,\left\langle A\mathrm{e}^{-iHt}\Psi,\ \Psi(t)\right\rangle,\\
G_2(T) &= -2\,\mathrm{Im}\int_0^T \mathrm{d}t\,\left\langle B\mathrm{e}^{-iHt}\Psi,\ \Psi(t)\right\rangle,\\
G_3(T) &= -2\,\mathrm{Im}\int_0^T \mathrm{d}t\,\left\langle B^*\mathrm{e}^{-iHt}\Psi,\ \Psi(t)\right\rangle.
\end{aligned}$$

It remains to bound these three terms.

3.6. BOUND ON G_1

If we write $A = \mathcal{N} + \tilde{A}$, we obtain from Lemma 4 that

$$\left\|\tilde{A}\xi\right\| \le C\left\|\left(\mathcal{N}+\alpha^{-2}\right)^{1/2}\xi\right\| \qquad \text{for all } \xi.$$

This allows us to bound

$$
\begin{aligned}
|G_1(T)| &\le 2 \int_0^T \mathrm{d}t \, \left(\|\mathcal{N}^{1/2} \mathrm{e}^{-iHt} \Psi\| \|\mathcal{N}^{1/2} \Psi(t)\| + \|\mathrm{e}^{-iHt} \Psi\| \|\tilde{A} \Psi(t)\| \right) \\
&\le 2 \int_0^T \mathrm{d}t \, \left(\|\mathcal{N}^{1/2} \mathrm{e}^{-iHt} \Psi\| \|\mathcal{N}^{1/2} \Psi(t)\| + CM \left\| \left(\mathcal{N} + \alpha^{-2}\right)^{1/2} \Psi(t) \right\| \right).
\end{aligned}
\tag{19}
$$

According to energy conservation, Lemma 8, we have

$$
\|\mathcal{N}^{1/2} \mathrm{e}^{-iHt} \Psi\| \le C \|(p^2 + \mathcal{N} + 1)^{1/2} \Psi\| \le CM.
$$

Thus, it remains to bound the norm of $\Psi(t)$ and $\mathcal{N}^{1/2}\Psi(t)$. By Lemma 4,

$$
\begin{aligned}
\|\Psi(t)\| &\le \|a^*(\mathrm{e}^{ikx} |k|^{-3} \chi_{\{|k|>\Lambda\}}) \mathrm{e}^{-iH_\varphi t} \Psi\| + \|a^*(\mathrm{e}^{ikx} |k|^{-3} \chi_{\{|k|>\Lambda\}}) \mathrm{e}^{-iH_\varphi T} \Psi\| \\
&\le 2 \left\| |k|^{-3} \chi_{\{|k|^{-3}>\Lambda\}} \right\| \left\| \left(\mathcal{N} + \alpha^{-2}\right)^{1/2} \Psi \right\| \\
&\le 2\sqrt{2} \left\| |k|^{-3} \chi_{\{|k|^{-3}>\Lambda\}} \right\| M\alpha^{-1},
\end{aligned}
$$

where we used (17). Moreover, again by Lemma 4,

$$
\begin{aligned}
\|\mathcal{N}^{1/2} \Psi(t)\| &\le \|\mathcal{N}^{1/2} a^*(\mathrm{e}^{ikx} |k|^{-3} \chi_{\{|k|>\Lambda\}}) \mathrm{e}^{-iH_\varphi t} \Psi\| \\
&\quad + \|\mathcal{N}^{1/2} a^*(\mathrm{e}^{ikx} |k|^{-3} \chi_{\{|k|>\Lambda\}}) \mathrm{e}^{-iH_\varphi T} \Psi\| \\
&\le 2 \left\| |k|^{-3} \chi_{\{|k|^{-3}>\Lambda\}} \right\| \left\| \left(\mathcal{N} + \alpha^{-2}\right) \Psi \right\| \\
&\le 4 \left\| |k|^{-3} \chi_{\{|k|^{-3}>\Lambda\}} \right\| M\alpha^{-2}.
\end{aligned}
$$

Note that the previous two bounds also imply that

$$
\left\| \left(\mathcal{N} + \alpha^{-2}\right)^{1/2} \Psi(t) \right\| \le C' M \alpha^{-2}.
$$

Inserting these bounds in (19) we infer that

$$
|G_1(T)| \le C'' M^2 \alpha^{-2} T,
$$

as required for Proposition 9.

3.7. BOUND ON G_2

Using the second inequality in Lemma 10, we get

$$|G_2(T)| \le 2\int_0^T \mathrm{d}t \left\| \left(\mathcal{N}+\alpha^{-2}\right)^{-1/2} B\mathrm{e}^{iHt}\Psi\right\| \left\| \left(\mathcal{N}+\alpha^{-2}\right)^{1/2} \Psi(t)\right\|$$
$$\le 2C\int_0^T \mathrm{d}t \left\| \left(p^2+1\right)^{1/2} \mathrm{e}^{iHt}\Psi\right\| \left\| \left(\mathcal{N}+\alpha^{-2}\right)^{1/2} \Psi(t)\right\|.$$

By energy conservation, Lemma 8, we have

$$\left\| \left(p^2+1\right)^{1/2} \mathrm{e}^{iHt}\Psi\right\| \le C\left\| \left(p^2+\mathcal{N}+1\right)^{1/2} \Psi\right\| \le CM.$$

This, together with the bound on $\left(\mathcal{N}+\alpha^{-2}\right)^{1/2}\Psi(t)$ that we derived when bounding G_1, yields a bound on G_2 of the desired form.

3.8. BOUND ON G_3

We bound, using the first inequality in Lemma 10,

$$|G_3(T)| \le 2\int_0^T \mathrm{d}t \left\|\mathrm{e}^{-iHt}\Psi\right\| \|B\Psi(t)\|$$
$$\le 2CM\int_0^T \mathrm{d}t \left\| \left(p^2+1\right)^{1/2} \mathcal{N}^{1/2}\Psi(t)\right\|.$$

By energy conservation, Lemma 3, together with Corollary 5 and the fact that $(1+|k|)|k|^{-3}\chi_{\{|k|>\Lambda\}} \in L^2$,

$$\begin{aligned}\left\| \left(p^2+1\right)^{1/2} \mathcal{N}^{1/2}\Psi(t)\right\| &\le \left\| \left(p^2+1\right)^{1/2} \mathcal{N}^{1/2} a^*(\mathrm{e}^{ikx}|k|^{-3}\chi_{\{|k|>\Lambda\}})\mathrm{e}^{-iH_\varphi t}\Psi\right\| \\ &\quad + C\left\| \left(p^2+1\right)^{1/2} \mathcal{N}^{1/2} a^*\left(\mathrm{e}^{ikx}|k|^{-3}\chi_{\{|k|>\Lambda\}}\right)\mathrm{e}^{-iH_\varphi T}\Psi\right\| \\ &\le C'\left\| \left(p^2+1\right)^{1/2} \left(\mathcal{N}+\alpha^{-2}\right)\mathrm{e}^{-iH_\varphi t}\Psi\right\| \\ &\quad + CC'\left\| \left(p^2+1\right)^{1/2} \left(\mathcal{N}+\alpha^{-2}\right)\mathrm{e}^{-iH_\varphi T}\Psi\right\| \\ &\le C''\left\| \left(p^2+1\right)^{1/2} \left(\mathcal{N}+\alpha^{-2}\right)\Psi\right\| \\ &\le 2C''M\alpha^{-2}.\end{aligned}$$

Again this shows that G_3 is bounded as needed for the application of Proposition 9. The proof of Proposition 9 is now complete.

Appendix A. Strong Coupling Units

In this appendix, we briefly explain how H_α^{F} is related to the more traditional form of the Fröhlich Hamiltonian

$$p^2+\sqrt{\alpha}\int_{\mathbb{R}^3}\frac{\mathrm{d}k}{|k|}\left(\mathrm{e}^{-ik\cdot x}a_k+\mathrm{e}^{ik\cdot x}a_k^*\right)+\int_{\mathbb{R}^3}\mathrm{d}k\,a_k^*a_k,$$

where now a_k^* and a_k satisfy

$$[a_k,a_{k'}^*]=\delta(k-k'),\qquad [a_k,a_{k'}]=[a_k^*,a_{k'}^*]=0\qquad \text{for all } k,\ k'\in\mathbb{R}^3.$$

Let $\tilde{x}=\alpha x$, so that $\tilde{p}=\alpha^{-1}p$. Then the above operator is unitarily equivalent to

$$\alpha^2\tilde{p}^2+\sqrt{\alpha}\int_{\mathbb{R}^3}\frac{\mathrm{d}k}{|k|}\left(\mathrm{e}^{-i\alpha^{-1}k\cdot\tilde{x}}a_k+\mathrm{e}^{i\alpha^{-1}k\cdot\tilde{x}}a_k^*\right)+\int_{\mathbb{R}^3}\mathrm{d}k\,a_k^*a_k.$$

By the change of variables $\tilde{k}=\alpha^{-1}k$, we can rewrite the operator as

$$\begin{aligned}
&\alpha^2\tilde{p}^2+\alpha^{5/2}\int_{\mathbb{R}^3}\frac{\mathrm{d}\tilde{k}}{|\tilde{k}|}\left(\mathrm{e}^{-i\tilde{k}\cdot\tilde{x}}a_{\alpha\tilde{k}}+\mathrm{e}^{i\tilde{k}\cdot\tilde{x}}a_{\alpha\tilde{k}}^*\right)+\alpha^3\int_{\mathbb{R}^3}\mathrm{d}k\,a_{\alpha\tilde{k}}^*a_{\alpha\tilde{k}}\\
&=\alpha^2\left(\tilde{p}^2+\int_{\mathbb{R}^3}\frac{\mathrm{d}\tilde{k}}{|\tilde{k}|}\left(\mathrm{e}^{-i\tilde{k}\cdot\tilde{x}}\left(\alpha^{1/2}a_{\alpha\tilde{k}}\right)+\mathrm{e}^{i\tilde{k}\cdot\tilde{x}}\left(\alpha^{1/2}a_{\alpha\tilde{k}}^*\right)\right)\right.\\
&\qquad\left.+\int_{\mathbb{R}^3}\mathrm{d}k\,\left(\alpha^{1/2}a_{\alpha\tilde{k}}\right)^*\left(\alpha^{1/2}a_{\alpha\tilde{k}}\right)\right).
\end{aligned}$$

Defining $\tilde{a}_{\tilde{k}}=\alpha^{1/2}a_{\alpha\tilde{k}}$, we find the commutation relations

$$[\tilde{a}_{\tilde{k}},\tilde{a}_{\tilde{k}'}^*]=\alpha^{-2}\delta(\tilde{k}-\tilde{k}'),\qquad [\tilde{a}_{\tilde{k}},\tilde{a}_{\tilde{k}'}]=[\tilde{a}_{\tilde{k}}^*,\tilde{a}_{\tilde{k}'}^*]=0\qquad \text{for all } \tilde{k},\ \tilde{k}'\in\mathbb{R}^3.$$

Thus, we have obtained the Hamiltonian $\alpha^2H_\alpha^{\mathrm{F}}$.

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